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Characterizability of Best Approximations by Means of a Kolmogorov Criterion

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Considering a space $C[Q, H]$ of continuous functions, supplied with the sup-norm, from a compact or not compact topological space Q into a unitary space H, an investigation is made of sufficient conditions for Q and H in order that best approximations can be characterized completely by means of a Kolmogorov criterion. The results obtained constitute an extension of earlier work by Brosowski for the case of compact metric Q and arbitrary unitary H.

1. INTRODUCTION

The approximation problem as treated by Brosowski [1] for spaces $C[Q, H]$ of continuous functions from a compact metric space Q into a unitary space H, and supplied with the sup-norm, seems to be one of the most general problems studied in the literature, where best approximations are characterized by means of a Kolmogorov criterion for the set of extremal points of the error function.

Let $G \subset C[Q, H]$ be the set of approximants, and $f \in C[Q, H]$, $f \notin G$. Then Brosowski's main result [1, p.78] states that a best approximation $s \in G$ to f can be characterized by means of a Kolmogorov criterion if and only if G satisfies some regularity property.

In this paper an attempt is made to extend this characterization theorem to the case that Q is an arbitrary topological space. Since now also non-compact spaces are involved, we extend Brosowski's regularity definition in the following way:

DEFINITION 1. *The set $G \subset C[Q, H]$ is called regular if for each pair $S, T \in G$, $\| T - S \| < \infty$, for each closed subset A of Q and for each function $F \in C[Q, H]$, $\| F \| < \infty$, the inequality*

$$\inf_{x \in A} \{\mathrm{Re}\langle F(x), (T - S)(x)\rangle\} > 0, \qquad (R_0)$$

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implies that for each $t \in \mathbf{R}_0^+$ there exists a $T_t \in G$ such that

$$\inf_{x \in A} \{2 \operatorname{Re}\langle F(x), (T_t - S)(x)\rangle - \|(S - T_t)(x)\|^2\} > 0, \tag{R_1}$$

and

$$\| S - T_t \| < t. \tag{R_2}$$

In this definition $\langle \cdot, \cdot \rangle$ denotes an inner product and $\operatorname{Re}\langle \cdot, \cdot \rangle$ its real part; $\| \cdot \|$ is the norm defined by the square root of an inner product.

Noncompactness of Q also implies that the error function $f - S$ can attain its supreme norm value in a subset of elements of Q which are not extremal points. Therefore we will use a Kolmogorov criterion on open set $O(e)$ defined by

$$O(e) = \{x \in Q: \|(f - S)(x)\| > \|f - S\| - e\}, \tag{1.1}$$

where e is an arbitrary positive number. Then, by using arguments similar to those of Brosowski it can be shown that the direct characterization theorem holds for arbitrary Q and arbitrary unitary spaces H, i.e., characterization by means of a Kolmogorov criterion is a consequence of the regularity of G in the sense of Definition 1. We state this as

THEOREM 1. *Let G_f be the set $\{T \in G: \|f - T\| < \infty\}$. If G is regular, then $S \in G$ is a best approximation to $f \in C[Q, H]$, $f \notin G$, iff for all $T \in G_f$ and all positive numbers e*:

$$\inf_{x \in O(e)} \{\operatorname{Re}\langle (f - S)(x), (T - S)(x)\rangle\} \leqslant 0. \quad \# \tag{1.2}$$

Following Brosowski, we will say that G has the *NS* property when the statement that S is a best approximation to f holds true iff S satisfies the Kolmogorov criterion (1.2).

It turns out that the inverse characterization theorem, i.e., the regularity of G being a consequence of the *NS* property, only holds true for certain classes of topological spaces Q and unitary spaces H. The keystone in the proof of the inverse theorem for compact metric Q is the construction of a function $g = f - S$ such that

$$g(x) = \frac{f_1(x)}{\|f_1(x)\|}, \qquad x \in A; \quad \| g(x)\| < 1, \quad x \in Q\backslash A, \tag{1.3}$$

where A is an arbitrary closed subset of Q, S and T are elements of G, and $f_1 \in C[Q, H]$ satisfies the inequality

$$\inf_{x \in A} \{\operatorname{Re}\langle f_1(x), (T - S)(x)\rangle\} > 0. \tag{1.4}$$

Since (1.4) is the negation of the classical Kolmogorov criterion, it follows that S is not a best approximation to f; then by standard arguments it is proved that G is regular. In extending the method for constructing g to the case of an arbitrary topological space Q, we must observe that (1.4) implies the existence of an open neighborhood A_0 of A such that

$$\inf_{x \in A_0} \{\mathrm{Re}\langle (f - S)(x), (T - S)(x)\rangle\} > 0. \tag{1.5}$$

But (1.5) not necessary implies the negation of the Kolmogorov criterion (1.2). To ensure that (1.5) implies the negation of (1.2) we must construct g in such a way that there exists a set $O(e)$, $e \in \mathbf{R}_0^+$, which satisfies $A \subset O(e) \subseteq A_0$.

Therefore we raise the following problem: Let there be given a closed subset of Q, and a function $f\colon Q \to H$, $\mathrm{def}(f) = A$, f continuous on A in the space Q. What conditions must be imposed on Q and H in order that there exists a function $g \in C[Q, H]$, $\mathrm{def}(g) = Q$, such that

(i) $g(x) = f(x) \quad \text{for} \quad x \in A;$ (1.6)

(ii) $\| g(x)\| < \sup_{x \in A} \| g(x)\| \quad \text{for} \quad x \in Q \backslash A;$ (1.7)

(iii) there exists a countable set $\{O_i\}$, $i \in I$, of nested open neighborhoods of A satisfying the conditions:

(a) $\bigcap_{i \in I} O_i = A; \qquad O_i \subseteq O_j \quad \text{for} \quad j < i;$ (1.8)

(b) $\sup_{x \in X_i} \| g(x)\| \leqslant \sup_{x \in X_j} \| g(x)\| < \sup_{x \in A} \| g(x)\|, \quad i < j,$ with $X_i = Q \backslash Q_i$; (1.9)

(c) for each open neighborhood A_0 of A there exists a set O_i such that $O_i \subseteq A_0$.

In the formulation of this problem the concept of a countable set of nested open sets is to be understood as follows:

Definition 2. A countable set of nested open sets $\{E_n : n \in I\}$ of a a topological space is a set such that for each pair n, m, $n < m$, $E_n \subseteq c(E_n) \subseteq E_m$, where c is the closure operator.

Urysohn proved the existence of such a function g in the case Q is compact metric, $H = \mathbf{R}$ (see Brosowski [1, p. 14]); Brosowski [1, p. 15] did the same for compact metric Q and arbitrary H, by using an extension of Tietze's theorem.

For solving the above problem, we will use some other extensions of Tietze's theorem (Section 2) and an extension of Urysohn's theorem (Section 4). Section 3 is devoted to the construction of open sets satisfying the conditions (iii). In the last section all results are assembled, leading to the final formulation of a generalized inverse characterization theorem. The meaning of symbols introduced here will be maintained throughout the paper.

The extension of Brosowski's characterization theorem, as presented here, is an ontgrowth of earlier work by the first author [2. Chap. 1].

2. Some Extensions of Tietze's Theorem

The purpose of this section is to find classes of topological spaces Q and unitary spaces H allowing for an extension $g_1 \in C[Q, H]$ of a given function $f: Q \to H$, $\operatorname{def}(f) = A$, with A a closed subset of Q, and f continuous on A in Q, which satisfies

$$g_1(x) = f(x) \qquad \text{for} \quad x \in A; \tag{2.1}$$

$$g_1(Q) \subseteq \text{convex hull of } f(A). \tag{2.2}$$

To this end we consider the classes N, CN, and M of normal topological spaces, collectionwise normal topological spaces (Dowker [3]) and metric spaces, respectively. Moreover we will use the following topological concepts.

Definition 3 (Hanner [4]). A topological space X is an extension space (ES) for a class W of topological spaces, if for each $Q \in W$, for each closed subset B of Q, and for each function $h: Q \to X$, $\operatorname{def}(h) = B$, h continuous on B in Q, there exists an extension $h^* \in C[Q, X]$ of h. We denote this as $X \in \mathrm{ES}(W)$.

Definition 4 (Hanner [4]). An absolute G_δ-space is a metric space which, whenever it is imbedded in a metric space, is a countable intersection of open sets.

Extension spaces for various classes W are studied in the theory of absolute retracts (Hanner [5, 6]). Hu [7] has shown that the class of absolute G_δ-spaces is equivalent with the class of complete metric spaces.

A first solutuion to the problem (2.1, 2.2) is provided by

Theorem 2 (Extension theorem of Tietze, see Pervin [8, p. 89]). *A topological space Q is normal iff for each closed subset $A \subseteq Q$ and each function $f: Q \to \mathbf{R}$, $\operatorname{def}(f) = A$, $f(A) \subseteq [a, b]$, f continuous on A in Q, there exists and extension $g_1 \in C[Q, \mathbf{R}]$ such that $g_1(x) = f(x)$ for $x \in A$, and $g_1(Q) \subseteq [a, b]$; i.e., $[a, b] \in ES(N)$.* #

A second solution arises as a special case of a theorem by Dugundji [9]:

THEOREM 3. *A unitary space and any convex subset of it is an* ES(M). #

A connection between the classes ES(N) and ES(M) is given by Dowker [3]:

LEMMA A. *Let Y be a metric space. Then $Y \in$* ES(N) *iff $Y \in$* ES(M) *and Y is a separable absolute G_δ-space (i.e., a separable complete metric space).* #

This lemma allows for a third solution to problem (2.1, 2.2).

THEOREM 4. *A separable complete unitary space H and each closed convex subset E of it belongs to* ES(N).

Proof. Since E is convex, Theorem 3 implies that $E \in$ ES(M). Further, E is complete (Kantorovich [10, p. 10]). On the other hand a separable metric space is second axiom (Pervin [8, p.10]) and a second axiom space is (hereditarily) separable (Pervin [8, p. 84]). Hence E is a separable complete metric space. Application of Lemma A then completes the proof. #

Since each metric space has a completion and the completion of a unitary space is a unitary space (Kantorovich [10, pp. 12, 80]), the condition of completeness imposed on H is a weak one. However, the condition for H to be separable is stronger, since the class of separable metric spaces is equivalent with the class of second axiom spaces (Pervin [8, p. 104]). We can get rid of this restriction upon H by means of

LEMMA B (Dowker [3]). *Let Y be a metric space. Then $Y \in$* ES(CN) *iff $Y \in$* ES(M) *and Y is an absolute G_δ-space.* #

As a corollary we then obtain

THEOREM 5. *A complete unitary space and each closed convex subset of it belongs to* ES(CN). #

We summarize all solutions found for problem (2.1, 2.2) in

THEOREM 6. *If* (i) $Q \in N$, $H = \mathbf{R}$ *or* (ii) $Q \in M$, *H unitary or* (iii) $Q \in N$, *H separable complete unitary or* (iv) $Q \in CN$, *H complete unitary, then for each $f: Q \to H$,* def(f) $= A$, *f continuous on A in the space Q, there exists an extension $g_1 \in C[Q, H]$ which satisfies the condition* (2.1) *and* (2.2). #

3. Construction of a Countable Set of Nested Open Neighborhoods of a Closed Set

For constructing a family $\{O_i\}$ of open sets satisfying the condition (iiia) of the problem stated in the Introduction, we will use the concept of a perfectly normal topological space (see Dowker [11]); a normal topological space Q is perfect if and only if each closed set of Q is the countable intersection of open sets. It is well-known that all metric spaces are perfectly normal. Poleunis [12] shows that regular second axiom spaces, regular first axiom paracompact spaces, and some other spaces are also perfectly normal.

THEOREM 7. *Let Q be perfectly normal and A_0 an open neighborhood of the closed subset A. Then there exists a countable family $\{O_i : i \in \mathbf{Z}\}$* of nested open neighborhoods of A such that

$$\bigcap_{i \in \mathbf{Z}} O_i = A; \qquad \bigcup_{i \in \mathbf{Z}} O_i \subseteq A_0 . \tag{3.1}$$

Proof. Since Q is normal, for each $m \in \mathbf{Z}$, there exist open sets X_m such that

$$A \subseteq \cdots \subseteq X_{m-1} \subseteq c(X_{m-1}) \subseteq X_m \subseteq c(X_m) \subseteq \cdots \subseteq A_0 ,$$

with c denoting the closure of a set (see Pervin [8, p. 88]. Hence the sequence $\{X_i , i \in \mathbf{Z}\}$ of nested open neighborhoods of A satisfies the second condition (3.1).

Since Q is perfectly normal there exists a sequence $\{V_i , i \in \mathbf{Z}^-\}$ of open neighborhoods of A, satisfying the first condition (3.1). With this sequence we define another, $\{P_i , i \in \mathbf{Z}^-\}$, as follows:

$$P_0 = V_0 ; \qquad P_0' \subseteq c(P_0') \subseteq V_0 ,$$

with P_0' an open neighborhood of A;

$$P_j = P_{j+1}' \cap V_j \qquad \text{with} \quad P_{j+1}' \subseteq c(P_{j+1}') \subseteq P_{j+1} , \quad j \in \mathbb{Z}_0^- .$$

Next define

$$O_i = X_i \cap P_i , \quad i \in \mathbf{Z}^-; \qquad O_i = X_i , \quad i \in \mathbf{Z}_0^+ .$$

Since

$$X_0 \cap P_0 \subseteq c(X_0 \cap P_0) \subseteq X_1$$

and

$$X_i \cap P_i \subseteq c(X_i \cap P_i) \subseteq c(X_i) \cap c(P_i) \subseteq X_{i+1} \cap P_{i+1} , \qquad i \in \mathbf{Z}_0^-$$

the family $\{O_i\}$ satisfies both conditions (3.1). #

By imposing additional conditions on Q, we let the sequence $\{O_i\}$ satisfy the condition that, for some i, $O_i \subseteq A_0$, A_0 being an arbitrary open neighborhood of A (see condition (iiic) in the Introduction). The topological concept we use here is aleph$_0$-compactness.

LEMMA C (Kowalsky [13, p. 87]. *A topological space is aleph$_0$-compact iff for each sequence $\{A_n, n \in \mathbf{N}\}$ of nonempty closed monotonically decreasing subsets of Q, $\bigcap_{n \in \mathbf{N}} A_n \neq \varnothing$.* #

Then we are able to prove

THEOREM 8. *Let Q be a perfectly normal aleph$_0$-compact topological space, and A a closed subset of it. Then there exists a countable family of nested open neighborhoods O_i of A such that, for each neighborhood A_0 of A*: (i) *the conditions* (3.1) *hold*; (ii) *there exists an element O_i of this family which is a subset of A_0*.

Proof. Consider the sequence $\{O_i, i \in \mathbf{Z}\}$, constructed in the proof of Theorem 7. Assume that for all $i \in \mathbf{Z}$ there exists an element z of Q such that $Z \in O_i \cap A_0^c$. Since

$$O_{i-1} \cap A_0^c \subseteq c(O_{i-1}) \cap A_0^c \subseteq O_i \cap A_0^c, \qquad i \in \mathbf{Z}$$

we have

$$\bigcap_{i \in \mathbf{Z}} \{c(O_i) \cap A_0^c\} = \bigcap_{i \in \mathbf{Z}} \{O_i \cap A_0^c\}.$$

Hence the sequence $\{W_i, i \in \mathbf{Z}\}$, $W_i = c(O_i) \cap A_0^c$ decreases monotonically for $i \to -\infty$, while each W_i is nonempty and closed. It then follows from Lemma C that $\bigcap_{i \in \mathbf{Z}} W_i \neq \varnothing$, contradicting statement (i). Hence the negation of statement (ii) is false. #

4. AN EXTENSION OF URYSOHN'S THEOREM

In this section we state and prove an extension of Urysohn's theorem under the conditions that Q is perfectly normal and aleph$_0$-compact, and $H = \mathbf{R}$. This is based on a definition and a lemma.

DEFINITION 5. We call the set of dyadic fractions $\mathbf{Q}^*$ the set of fractions belonging to $]0, 1[$ and whose denominates are powers of 2.

LEMMA D. *Let Q be perfectly normal and aleph$_0$-compact, A_1 and A_2 disjoint closed subsets of Q, and $A_{1,1}$ an open neighborhood of A_1. Then there exist totally ordered sets $V_1 = \{O_{1,t}, t \in M_1\}$, $M_1 = \mathbf{Q}^* \cap\,]0, \frac{1}{2}[$ and*

$V_2 = \{O_{2,t}, t \in M_2\}$, $M_2 = \mathbf{Q}^* \cap]\frac{1}{2}, 1[$, *of nested open neighborhoods of* A_1 *and* A_2, *respectively, such that*

(i) *if* $(-1)^{j+1} t_1 < (-1)^{j+1} t_2$ *then* $c(O_{j,t_1}) \subseteq c(O_{j,t_2})$, $(t_1, t_2) \in M_j \times M_j$, $j = 1, 2$;

(ii) $A_j = \bigcap_{t \in M_j} O_{j,t}$, $j = 1, 2$; $\bigcup_{t \in M_1} O_{1,t} \subseteq A_{11}$; $\bigcup_{t \in M_2} O_{2,t} \subseteq i(A_{1,1}^c)$;

(iii) *for each arbitrary open neighborhood of* A_j *there exists an element* $O_{j,t}$ *of* V_j *which is a subset of it* ($j = 1, 2$).

Proof. Theorem 8 guarantees the existence of families $U_j = \{O_{j,i}, i \in \mathbf{Z}\}$, $j = 1, 2$, satisfying the conditions (ii) and (iii) for $M_j = \mathbf{Z}$. In order to construct V_1 out of the family U_1 we define a mapping from $\mathbf{Q}^* \cap]0, \frac{1}{2}[$ to U_1 as follows:

$$2^{-|j|-2} \to O_{1,j}, \quad j \in \mathbf{Z}^-; \qquad 2^{-1} - 2^{-j-2} \to O_{1,j}, \quad j \in \mathbf{Z}_0^+.$$

Consider the element $O_{1,0} \in U_j$. Then $A_1 \subseteq O_{1,0} \subseteq c(O_{1,0}) \subseteq A_{1,1}$. This chain of inclusions can be enlarged to

$$\begin{aligned} A_1 \subseteq O_{1,-1} \subseteq c(O_{1,-1}) \subseteq O_{-1}^* \subseteq c(O_{-1}^*) \subseteq O_{1,0} \subseteq c(O_{1,0}) \\ \subseteq O_1^* \subseteq c(O_1^*) \subseteq O_{1,1} \subseteq A_{1,1}, \end{aligned} \tag{3.2}$$

where the existence of the open sets O_{-1}^*, O_1^* is guaranteed by the normality of Q; we consider them as the images of the elements $\frac{3}{16}$ and $\frac{5}{16}$, respectively, of $\mathbf{Q}^* \cap]0, \frac{1}{2}[$, i.e., half the sum of the elements which are mapped onto the adjacent open sets $O_{1,-1}$, $O_{1,0}$ and $O_{1,0}$, $O_{1,1}$, respectively.

The chain (3.2) can be enlarged successively by adding sets $O_{1,-j}$ and $O_{1,j}$ to the left and the right, respectively, $j = 2, 3, \ldots$, and by inserting again and again an open set between each two adjacent open sets already present, while considering it as an image of an element of $\mathbf{Q}^* \cap]0, \frac{1}{2}[$ being the half-sum of the elements mapped onto these adjacent sets. It is easy to see that we obtain in this manner a totally ordered family V_1 which satisfies the conditions (i)–(iii). The family V_2 can be constructed in a completely similar way. #

THEOREM 9 (Extension of Urysohn's theorem). *Let the assumptions Lemma D hold. Assume further that* $f: Q \to \mathbf{R}$, $\operatorname{def}(f) = A_1 \cup A_2$, $f(A_1) = \{0\}$, $f(A_2) = \{1\}$, *and* f *continuous on* $A_1 \cup A_2$ *in the space* Q. *Let, for each* $(t_1, t_2) \in M_1 \times M_2$, $O(t_1, t_2)$ *be the set* $Q \backslash (O_{1,t_1} \cup O_{2,t_1})$. *Then there exists an extension* $g \in C[Q, \mathbf{R}]$ *of* f *which satisfies the following conditions*:

(i) $g(x) = f(x)$, $x \in A_1 \cup A_2$;

(ii) $\inf_{x \in O_{2,t}} g(x) \geqslant t$ *for* $t \in M_2$ *and* $\sup_{x \in O_{1,t}} g(x) \leqslant t$ *for* $t \in M_1$;

(iii) $\sup_{x\in O(t_1,t_2)} g(x) \leqslant t_2$ *and* $\inf_{x\in O(t_1,t_2)} g(x) \geqslant t_1$ *for* $(t_1, t_2) \in M_1 \times M_2$;

(iv) *for each* $t_1 < u_1$, $t_2 > u_2$, $(t_1, t_2), (u_1, u_2) \in M_1 \times M_2$:

$$0 < \inf_{x\in O(t_1,t_2)} g(x) \leqslant \inf_{x\in O(u_1,u_2)} g(x) \leqslant \sup_{x\in O(u_1,u_2)} g(x) \leqslant \sup_{x\in O(t_1,t_2)} g(x) < 1.$$

Proof. We define a function $g(x)$ as

$$\begin{aligned} g(x) &= \inf_{x\in O_{1,t}} t, \qquad x \in A_{1,1}, \\ &= \sup_{x\in O_{2,t}} t, \qquad x \in i(A_{1,1}^c), \\ &= 2^{-1}, \qquad x \in Q\backslash(A_{1,1} \cup i(A_{1,1}^c)). \end{aligned} \tag{3.3}$$

Since $U_1 \subset V_1$ and $U_2 \subset V_2$ it is readily seen from (3.3) that g satisfies condition (i).

Now we prove that $g \in C[Q, \mathbf{R}]$. This can be achieved by showing that $g^{-1}([0, p[)$ and $g^{-1}(]q, 1])$, $p, q \in]0, 1[$, are open in Q (see Lipschutz [14, p. 103]). Since M_1 is dense in $[0, \frac{1}{2}]$, there exists, for a number $p_1 \leqslant \frac{1}{2}$ and an $x \in g^{-1}([0, p_1[)$, a $t_x \in M_1$ such that $g(x) < t_x < p_1$. From the definition of $g(x)$ it then follows that $x \in O_{1,t_x}$, and hence

$$g^{-1}([0, p_1[) \subseteq \bigcup_{\substack{t<p_1\\ t\in M_1}} O_{1,t}.$$

To prove the inverse inclusion, consider an element y of the right member set. Then there exists a $t_y \in M_1$ such that $t_y < p_1$, $y \in O_{1,t_y}$ and hence $y \in g^{-1}([O, p_1[)$. This proves that

$$g^{-1}([0, p_1[) = \bigcup_{\substack{t<p_1\\ t\in M_1}} O_{1,t}. \tag{3.3}$$

It can be shown in an analogous way that, for $q_1 \geqslant \frac{1}{2}$:

$$g^{-1}(]q_1, 1]) = \bigcup_{\substack{t>q_1\\ t\in M_2}} O_{2,t}. \tag{3.5}$$

It is evident that the sets given by (3.4) and (3.5) are both open for $p_1 \leqslant \frac{1}{2}$, $q_1 \geqslant \frac{1}{2}$. We prove that $g^{-1}([0, p_1[)$ and $g^{-1}(]q_1, 1])$ for $p_1 > \frac{1}{2}$ and $q_1 < \frac{1}{2}$, respectively, are also open. Consider the set

$$P = \bigcup_{\substack{t>q_1\\ t\in M_1}} [c(O_{1,t})]^c.$$

For an x such that $g(x) \in]q_1, 1[$ there are elements t_1, t_2 of M_1 such that $q_1 < t_1 < t_2 < g(x)$. Hence $x \notin O_{1,t_2}$. Since $t_1 < t_2$ we have $O_{1,t_1} \subseteq c(O_{1,t_1}) \subseteq O_{1,t_2}$; consequently $x \in [c(O_{1,t_1})]^c$ and $g^{-1}(]q_1, 1]) \subseteq P$. To prove the inverse inclusion, consider a $y \in P$. Then there exists a $t_y \in M_1$ such that $t_y > q_1$ and $y \in [c(O_{1,t_y})]^c$. Furthermore, for $t < t_y$ we have $O_{1,t} \subseteq O_{1,t_y} \subseteq c(O_{1,t_y})$ and hence $y \notin O_{1,t}$. Thus $g(y) \geqslant t_y > q_1$ and $y \in g^{-1}(]q_1, 1])$. This proves that $g^{-1}(]q_1, 1]) = P$. By similar arguments

$$g^{-1}([0, p_1[) = \bigcup_{\substack{t<p_1 \\ t\in M_2}} [c(O_{2,t})]^c.$$

From this it follows that $g^{-1}([0, p_1[)$ and $g^{-1}(]q_1, 1])$ for $p_1 > \frac{1}{2}$, $q_1 < \frac{1}{2}$, respectively, are both open. Consequently $g(x)$ is continuous.

Next, we prove statement (iii). Assume that $r = \inf_{x\in O_{2,t}} g(x) < t$. Then there exists a $t_1 \in M_2$ such that $r < t_1 < t$ and hence an $x \in O_{2,t}$ such that $x \notin O_{2,t_1}$, contradicting the fact that $O_{2,t} \subseteq O_{2,t_1}$. The second inequality (ii) is proved similarly.

To prove statement (ii), assume that $r = \sup_{x\in O(t_1,t_2)} g(x) > t_2$. Then there exists a $u \in M_2$ such that $r > u > t_2$ and hence an $x \in O(t_1, t_2)$ such that $g(x) \geqslant u$. Hence $x \in O_{2,u}$, which is impossible since $O_{2,u} \subseteq O_{2,t_2}$. The other inequality (iii) and statement (iv) are proved in a similar way. #

5. A Generalized Inverse Characterization Theorem

The two extension Theorems, 6 and 9, lead us to consider the following classes of function spaces:

$C_1 = \{C[Q, H]$, Q compact metric and H arbitrary unitary$\}$,

$C_2 = \{C[, H]$, Q perfectly normal, aleph$_0$-compact, and H separable complete unitary$\}$,

$C_3 = \{C[Q, H]$, Q collectionwise perfectly normal, aleph$_0$-compact, and H complete unitary$\}$.

The class C_1 results from case (ii) of Theorem 6 and the fact that aleph$_0$-compact second axiom spaces (including metric spaces) are compact; this class was treated at length by Brosowski [1].

For these classes the following more general extension theorem holds:

Lemma E. *Let $C[Q, H]$ belong to one of the classes C_1, C_2, or C_3. Then*:

(i) *for each closed subset A of Q there exists a countable ordered family $V_2 = \{O_{2,t}, t \in M_2\}$, possessing the properties formulated in Lemma D*;

(ii) *each function* $f: Q \to H$, $\operatorname{def}(f) = A$ *has an extension* $g \in C[Q, H]$ *such that*

(a) $g(x) = f(x), x \in A$,

(b) $\| g(x)\| < \sup_{x\in A} \| g(x)\|, x \in Q\backslash A$,

(c) *for* $t < u$, $t, u \in M_2$, *and* $O(t) \equiv Q\backslash O_{2,t}$:

$$0 < \sup_{x\in O(t)} \| g(x)\| \leqslant \sup_{x\in O(u)} \| g(x)\| < \sup_{x\in A} \| g(x)\|. \quad \#$$

Now we are in a position to prove the following generalized inverse characterization theorem:

THEOREM 10. *Let* $C[Q, H]$ *belong to one of the classes* C_1, C_2, *or* C_3, *and* $G \subset C[Q, H]$. *If an element* $S \in G$ *is a best approximation to a function* $f \in C[Q, H], f \notin G$, *if and only if for all* $T \in G_f$,

$$\inf_{x\in E(f,S)} \{\operatorname{Re}\langle (f - S)(x), (T - S)(x)\rangle\} \leqslant 0, \tag{5.1}$$

with $E(f, S)$ *the set of extremal points of* $f - S$ *in* Q, *then* G *is regular* (*in the sense of Definition* 1).

Proof. Since Q is aleph$_0$-compact, Q is countably compact (i.e., every infinite set of Q has at least one limit point, Kowalsky [13, p.82]). As a consequence, the generalized Kolmogorov criterion is equivalent to the classical one.

Consider an arbitrary element $f_1 \in C[Q, H]$, $\| f_1 \| < \infty$, $S, T \in G_{f_1}$, and an arbitrary closed subset A of Q, for which statement (R_0) holds true. Then, by virtue of the Cauchy–Schwartz inequality, there exist numbers $a, a_1 \in \mathbf{R}_0{}^+$ such that

$$\inf_{x\in A} \| f_1(x)\| \geqslant a; \qquad \inf_{x\in A} \|(S - T)(x)\| \geqslant a_1 .$$

Now Lemma E implies the existence of a function $g \in C[Q, H]$ such that $g(x) = f_1(x)/\| f_1(x)\|$ for $x \in A$, and satisfying the conditions (iib, c). When introducing the function

$$f = \tfrac{1}{2}\min(a, t)\, g + S,$$

condition (R_0) implies

$$\inf_{x\in A} \left\{\operatorname{Re} \left\langle \frac{f_1(x)}{\| f_1(x)\|}, (T - S)(x)\right\rangle\right\} > 0.$$

Since the inner product is a continuous function, there consequently exists an open neighborhood A_0 of A such that

$$\inf_{x \in A_0} \{\mathrm{Re}\langle (f - S)(x), (T - S)(x) \rangle\} > e_1 , \tag{5.2}$$

with e_1 arbitrary small positive.

The sets $O(e) = \{x \in Q: \| g(x) \| > 1 - e\}$, $e \in \mathbf{R}_0{}^+$, are open in Q. From Lemma D we infer that there exists a set O_{2,t_1}, $t_1 < 1$, such that $O_{2,t_1} \subsetneq A_0$. Then it follows from (iic) of Lemma E that

$$\sup_{x \in O(t_1)} \| g(x) \| \leqslant t_1 .$$

Consequently there exists an $e \in]0, 1 - t_1[$ such that $A \subsetneq O(e) \subsetneq O_{2,t_1} \subsetneq A_0$. Hence it follows from (5.2) that

$$\inf_{x \in O(e)} \{\mathrm{Re}\langle (f - S)(x), (T - S)(x) \rangle\} > 0.$$

The assumptions of the theorem then imply that S is not a best approximation f. The conclusion that G is regular now follows by exactly the same arguments as used by Brosowski [1, p.78]. #

We recall that a generalized direct characterization is given as Theorem 1 and holds under no restrictions at all upon Q and H.

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